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On Non-Fock Boson Stochastic Integrals

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We show that certain non-Fock quantum boson stochastic integrals are canonically defined as closed operators. We also present an alternative formulation of the martingale representation theorem of Hudson and Lindsay (*J. Funct. Anal.* **61** (1985), 202–221).

INTRODUCTION

We consider non-Fock quantum boson stochastic integrals. These were constructed in [1] for certain quasi-free representations of the CCR and, for suitable integrands, were shown to define closed operators affiliated to the appropriate algebra. We show that these operators have a "canonical" definition. This was not apparent from the discussion in [1], where it appeared that the definition depended on the support of the integrand.

We also present an alternative formulation of the martingale representation theorem of [2]. The theory is expressed naturally in terms of vector-valued processes in the Sobolev space $\mathscr{H}^{+1}$ associated with the modular operator of the cyclic quasi-free representation under consideration. We obtain a slight generalization of the result in [2] in as much as we do not require the representation to be determined by a constant.

The relationship between [2] and Sections 5 and 6 of [1] becomes apparent if one observes that eqn. (2.19) of [2] implies that $\psi \in \operatorname{Dom} S = \mathscr{H}^{+1}$ and $S\psi = \psi^{\dagger}$, where S is the conjugation operator, and the correspondence $\psi \leftrightarrow T$ of Eqs. (2.20), (2.21) is just as in Sections 4 and 6 of [1]. The

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norm in [2, Eq. (6.1)] is that of Definition 6.7 of [1] with $\alpha_1 = 1$, $\alpha_2 = 0$, $dy = ds$, and with $\mu = \tau$, $\lambda = (1+\tau)^{1/2}$, and the norm on the set of martingales in [2] is just the $\mathscr{H}^{+1}$ norm as given in [1].

1. Non-Fock Quantum Boson Stochastic Integrals

We shall consider the CCR in the representation determined by the gauge-invariant quasi-free state given by

$$\omega(a^*(f)\, a(g)) = \int_0^\infty f(s)\, \overline{g(s)}\, \tau(s)\, ds$$

for $f, g \in D(T^{1/2})$, where $\tau \in L^\infty_{\rm loc}(\mathbb{R}^+)$, $\tau > 0$ a.e., and T is the operator "multiplication by τ" in $L^2(\mathbb{R}^+)$. We realize the creation and annihilation operators and the Weyl operators concretely as operators on the tensor product of two copies of the symmetric Fock space over $L^2(\mathbb{R}^+)$ as in [1]. Explicitly,

$$a^*(f) = a_0^*((1+T)^{1/2} f) \otimes \mathbb{1} + \mathbb{1} \otimes a_0(T^{1/2} \bar{f})$$
$$a(g) = a_0((1+T)^{1/2} g) \otimes \mathbb{1} + \mathbb{1} \otimes a_0^*(T^{1/2} \bar{g})$$

where $a_0^*(\cdot)$ and $a_0(\cdot)$ are the usual creation and annihilation operators on Fock space over $L^2(\mathbb{R}^+)$. The Weyl operators are given by

$$W(f) = \exp(a^*(f) - a(f))^-$$

as in [2].

For $t \geqslant 0$, $\mathfrak{P}_t$ denotes the polynomial *-algebra generated by the $a^*(f)$, $a(g)$, where f, g run over $L^2(\mathbb{R}^+)$ with support in $[0, t]$. We denote by $\mathscr{A}_t$ the C^*-algebra generated by all Weyl operators $W(f)$ where f has support in $[0, t]$.

The state ω is given by the vector $\Omega = \Omega_0 \otimes \Omega_0$, where Ω_0 is the Fock vacuum vector. Let $\mathscr{H}_t$ be the subspace generated by $\mathfrak{P}_t \Omega$ (equivalently, by $\mathscr{A}_t \Omega$), and set $\mathfrak{P} = \bigcup_{t \geqslant 0} \mathfrak{P}_t$, $\mathscr{H} = \overline{\mathfrak{P}\Omega} = \overline{\bigcup_{t \geqslant 0} \mathscr{H}_t}$, and let $\mathscr{A}$ denote the C^*-algebra generated by $\bigcup_{t \geqslant 0} \mathscr{A}_t$.

Since $\tau > 0$ almost everywhere, it follows that Ω is cyclic and separating for $\mathscr{A}''$ on $\mathscr{H}$. Let $S = J\Delta^{1/2}$ be the conjugation operator determined by the vector state Ω on the von Neumann algebra $\mathscr{A}''$. Denote by $\mathscr{H}^{+1}$ the Hilbert space $D(S) = D(\Delta^{1/2})$ with norm given by

$$\|\xi\|_{+1}^2 = \|\xi\|^2 + \|\Delta^{1/2}\xi\|^2$$

for $\xi \in D(S)$.

It was shown in [1] that the projections $P_t\colon \mathscr{H} \to \mathscr{H}_t$ have the property that $P_t\colon \mathscr{H}^{+1} \to \mathscr{H}^{+1}$ and $s \mapsto P_s$ is strongly continuous on $\mathscr{H}^{+1}$. These projections determine a filtration with respect to which one can define (adapted) processes, martingales, etc., as in [1]. For example, $(\zeta_t)_{t \geqslant 0}$ is an $\mathscr{H}^{+1}$-valued martingale if (ζ_t) is a process, i.e., $\zeta_t \in \mathscr{H}^{+1} \cap \mathscr{H}_t = \mathscr{H}_t^{+1}$, $t \geqslant 0$, and $P_s \zeta_t = \zeta_s$ for $0 \leqslant s \leqslant t$.

For given $u \in L^2_{\text{loc}}(\mathbb{R}^+)$, $\alpha_1, \alpha_2 \in \mathbb{C}$, set $A_t(u) = a(\chi_{[0,t]}u)$, $A_t^*(u) = a^*(\chi_{[0,t]}u)$ and $Y_t = \alpha_1 A_t^*(u) + \alpha_2 A_t(u)$ for $t \geqslant 0$. Let $\mathfrak{R}^{\text{loc}}_{\tau,\Delta}(\mathbb{R}^+, |u(s)|^2 ds; \mathscr{H}^{+1})$ denote the set of (classes of) $\mathscr{H}^{+1}$-valued processes ξ on $\mathbb{R}^+$ such that

$$\int_0^t \|\xi(s)\|^2_{\tau(s),\Delta} |u(s)|^2 ds < \infty$$

for all $t \geqslant 0$, where

$$\|\xi(s)\|^2_{\tau(s),\Delta} = \{|\alpha_1|^2(1+\tau(s)) + |\alpha_2|^2 \tau(s)\} \|\xi(s)\|^2 + \{|\alpha_1|^2 \tau(s) + |\alpha_2|^2(1+\tau(s))\|\Delta^{1/2}\xi(s)\|^2.$$

Then it was shown in [1] that the quantum stochastic integral $\zeta_t = \int_0^t \xi(s)\, dY_s$ defines an $\mathscr{H}^{+1}$-valued centered martingale.

Furthermore, for each $t \geqslant 0$, it was shown that the map $T_{\zeta_t}\colon x'\Omega \mapsto x'\zeta_t$, $x' \in \mathscr{A}'_t$, is closable and its closure is affiliated to $\mathscr{A}''_t$. Thus we are able to consider the quantum stochastic integral $\int_0^t \xi(s)\, dY_s$ as a closed operator affiliated to $\mathscr{A}''_t$.

However, suppose that $\xi(\cdot)$ vanishes on $[r, t]$ for some $0 \leqslant r < t$. Then

$$\zeta_t = \int_0^t \xi(s)\, dY_s = \int_0^r \xi(s)\, dY_s = \zeta_r$$

and so we can define two operators T_{ζ_t}, as above, and T_{ζ_r} given by $T_{\zeta_r}\colon y'\Omega \mapsto y'\zeta_r$, $y' \in \mathscr{A}'_r$. Since $\mathscr{A}_r \subset \mathscr{A}_t$, we have $\mathscr{A}'_t \subset \mathscr{A}'_r$ and so it is clear that T_{ζ_r} is an extension of T_{ζ_t}. Indeed, we can consider ζ_r as an element of $\mathscr{H}_t^{+1}$ for any $t \geqslant r$ and thus obtain a family $\{\bar{T}_{\zeta_t}\colon t \geqslant r\}$ of closed operators with the properties that $\bar{T}_{\zeta_t}$ is affiliated to $\mathscr{A}''_t$, $\bar{T}_{\zeta_{t_1}} \supseteq \bar{T}_{\zeta_{t_2}}$ for $r \leqslant t_1 \leqslant t_2$, and each $\bar{T}_{\zeta_t}$ is "naturally" associated with the stochastic integral $\int_0^t \xi(s)\, dY_s$.

We shall show that this ambiguity is only apparent; all the T_{ζ_t}'s have the *same* closure.

2. Uniqueness of the Stochastic Integral Operator

The proof of the uniqueness rests on the product structure of the representation of the CCR and the following lemma.

LEMMA 2.1. *Let $\mathcal{M}$, $\mathcal{N}$ be von Neuman algebras acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, with cyclic and separating vectors Ω_1 and Ω_2. For given $\Phi \in \mathcal{H}_1$, define operators L and T on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by L: $x'\Omega_1 \otimes \Omega_2 \mapsto x'\Phi \otimes \Omega_2$, $x' \in (\mathcal{M} \otimes \mathcal{N})'$ and T: $y'\Omega_1 \otimes \Omega_2 \mapsto y'\Phi \otimes \Omega_2$, $y' \in (\mathcal{M} \otimes \mathbb{1})'$. Suppose, further, that L is closeable. Then $T \subseteq \bar{L}$.*

Proof. Let $\eta \in (\mathcal{M} \otimes \mathbb{1})'\Omega_1 \otimes \Omega_2 = \mathcal{M}' \otimes \mathcal{B}(\mathcal{H}_2)\, \Omega_1 \otimes \Omega_2$, thus $\eta = y'\Omega$, where $\Omega = \Omega_1 \otimes \Omega_2$ and $y' \in \mathcal{M}' \otimes \mathcal{B}(\mathcal{H}_2)$. Now, $\mathcal{M}' \otimes_{\text{alg}} \mathcal{B}(\mathcal{H}_2)$ is strongly dense in $\mathcal{M}' \otimes \mathcal{B}(\mathcal{H}_2)$ and so there is a sequence (y'_n) in $\mathcal{M}' \otimes_{\text{alg}} \mathcal{B}(\mathcal{H}_2)$ such that $y'_n\Omega \to \eta = y'\Omega$ and $y'_n\Phi \otimes \Omega_2 \to y'\Phi \otimes \Omega_2 = Ty'\Omega = T\eta$.

For each n, we can write y'_n as

$$y'_n = \sum_{i=1}^{k_n} a'_i \otimes z_i$$

for some k_n and $a'_i \in \mathcal{M}'$, $z_i \in \mathcal{B}(\mathcal{H}_2)$, $1 \leqslant i \leqslant k_n$. Since Ω_2 is cyclic for $\mathcal{N}'$, there is a sequence $(z_i^{(m)})$ in $\mathcal{N}'$ such that $z_i^{(m)}\Omega_2 \to z_i\Omega_2$ in $\mathcal{H}_2$. Putting

$$y_n'^{(m)} = \sum_{i=1}^{k_n} a'_i \otimes z_i^{(m)}, \qquad m = 1, 2, \ldots$$

we have that $y_n'^{(m)} \in \mathcal{M}' \otimes_{\text{alg}} \mathcal{N}'$ and $y_n'^{(m)}\Omega \to y'_n\Omega$ as $m \to \infty$.

But $y_n'^{(m)}\Omega \in \text{Dom}\, L$ and

$$\begin{aligned} Ly_n'^{(m)}\Omega &= \sum_{i=1}^{k_n} a'_i\Phi \otimes z_i^{(m)}\Omega_2 \\ &\to \sum_{i=1}^{k_n} a'_i\Phi \otimes z_i\Omega_2 \qquad \text{as} \quad m \to \infty \\ &= Ty'_n\Omega. \end{aligned}$$

We deduce that $y'_n\Omega \in \text{Dom}\, \bar{L}$ and $\bar{L}y'_n\Omega = Ty'_n\Omega$. Now, $y'_n\Omega \to \eta$ and $Ty'_n\Omega = y'_n\Phi \otimes \Omega_2 \to Ty'\Omega = T\eta$, and so we deduce that $\eta \in \text{Dom}\, \bar{L}$ and $\bar{L}\eta = T\eta$.

It follows that $T \subseteq \bar{L}$. Q.E.D.

THEOREM 2.2 *Let $\zeta \in \mathcal{H}_t^{+1}$, $t \geqslant 0$, and define the operators L_ζ and T_ζ by L_ζ: $x'\Omega \mapsto x'\zeta$, $x' \in \mathcal{A}'$, and T_ζ: $y'\Omega \mapsto y'\zeta$, $y' \in \mathcal{A}'_t$. Then $\bar{T}_\zeta = \bar{L}_\zeta$.*

Proof. Evidently, $L_\zeta \subseteq T_\zeta$, and, since $\zeta \in \mathcal{H}^{+1}$, one sees that both L_ζ and T_ζ are closeable [1].

Now, it is well-known that corresponding to the decomposition of $L^2(\mathbb{R}^+)$ as $L^2([0, t]) \oplus L^2([t, \infty])$ one can write $\mathcal{H}$ as $\mathcal{H}_t \otimes \mathcal{H}^t$, $\mathcal{A}$ as $\hat{\mathcal{A}}_t \otimes \hat{\mathcal{A}}^t$ and Ω as $\Omega_t \otimes \Omega^t$, where Ω_t (resp. Ω^t) is cyclic and separating for $\hat{\mathcal{A}}_t$ (resp. $\hat{\mathcal{A}}^t$) on $\mathcal{H}_t$ (resp. $\mathcal{H}^t$) and where $\hat{\mathcal{A}}_t$ (resp. $\hat{\mathcal{A}}^t$) is the von

Neumann algebra on $\mathscr{H}_t$ (resp. $\mathscr{H}^t$) generated by the quasi-free Weyl operators with test-functions in $L^2([0, t])$ (resp. $L^2([t, \infty)))$.

Under this identification, $\mathscr{A}_t$ is identified with $\hat{\mathscr{A}}_t \otimes \mathbb{1}$ and ζ is identified as an element of $\mathscr{H}_t$.

We can now apply lemma 2.1 to deduce that $T_\zeta \subseteq \bar{L}_\zeta$. It follows then that $L_\zeta \subseteq T_\zeta \subseteq \bar{L}_\zeta$ and so we see that $\bar{T}_\zeta = \bar{L}_\zeta$. Q.E.D.

Thus, we see that if $\zeta \in \mathfrak{K}^{\mathrm{loc}}_{\tau,\Delta}$ and ζ is the stochastic integral of section 1, $\zeta = \int_0^t \xi(s)\, dY_s$, then $\bar{T}_\zeta = \bar{L}_\zeta$ and it is irrelevant whether we think of ζ as an element of $\mathscr{H}_t^{+1}$ or $\mathscr{H}_s^{+1}$ for any $s \geqslant t$; i.e., the *construction* of $\bar{T}_\zeta$ is independent of the support of ξ. (Of course, the *action* of $\bar{T}_\zeta$ depends on ζ and therefore on ξ and hence its support.)

3. Martingale Representation Theorem

We will write a_t for $a(\chi_{[0,t]})$ and a_t^* for $a^*(\chi_{[0,t]})$. Furthermore, we will suppose that τ is locally bounded away from zero, i.e., we assume that for each $t > 0$ there is $\delta_t > 0$ such that $\tau \geqslant \delta_t$ on $[0, t]$. Then [1], for any process ξ in L^2_{loc} $(\mathbb{R}^+, ds;\ \mathscr{H}^{+1})$, the quantum stochastic integrals $\int_0^t \xi(s)\, da_s$ and $\int_0^t \xi(s)\, da_s^*$ are defined and determine centered $\mathscr{H}^{+1}$-valued martingales. We wish to establish the converse result.

For $t > 0$, let $\mathscr{K}_t$ denote the Hilbert space of (equivalence classes of) $\mathscr{H}^{+1}$-valued processes ϕ on $[0, t]$ equiped with the norm given by

$$\|\phi\|^2 = \int_0^t \{(1 + \tau(s))\|\phi(s)\|^2 + \tau(s)\|S\phi(s)\|^2\}\, ds.$$

Note that, since τ is locally bounded away from zero, $\phi \in \mathscr{K}_t$ implies that $\|S\phi\| < \infty$ and so $\int_0^t \phi(s)\, da_s^\#$ and $\int_0^t S\phi(s)\, da_s^\#$ are well-defined; $a_s^\#$ denotes either a_s or a_s^*.

THEOREM 3.1. *Let (ζ_t) be an $\mathscr{H}^{+1}$-valued martingale. Then there is a unique $\alpha \in \mathbb{C}$ and unique processes ϕ, ψ in $L^2_{\mathrm{loc}}(\mathbb{R}^+, ds;\ \mathscr{H}^{+1})$ such that*

$$\zeta_t = \alpha\Omega + \int_0^t \phi(s)\, da_s^* + S\psi(s)\, da_s$$

for $t \geqslant 0$.

As in [2], the proof is by a number of lemmas, the first of which is the key observation.

LEMMA 3.2. *For $f \in L^2_{\rm loc}(\mathbb{R}^+)$, set $W_f(t) = e^{\|(1+2\tau^2)^{1/2} f\chi_{[0,t]}\|^2/2}\, W(f\chi_{[0,t]})$. Then when $W_f(t)\,\Omega$ satisfies the stochastic differential equation*

$$d\zeta = f\zeta\, da^* - \bar{f}\zeta\, da$$

with $\zeta_0 = \Omega$; i.e.,

$$W_f(t)\,\Omega = \Omega + \int_0^t f(s)\, W_f(s)\,\Omega\, da_s^* - \int_0^t \bar{f}(s)\, W_f(s)\,\Omega\, da_s.$$

In particular, $(W_f(t)\,\Omega)_{t \geqslant 0}$ is an $\mathscr{H}^{+1}$-valued martingale.

Proof. Clearly $W_f(t)\,\Omega \in \mathscr{H}^{+1}$. That $W_f(t)\,\Omega$ satisfies the stochastic differential equation is established as in [2].

LEMMA 3.3. *Suppose that $\eta \in \mathscr{H}_t^{+1}$ and is orthogonal (in $\mathscr{H}^{+1}$) to $W_f(t)\,\Omega$ for every $f \in L^2_{\rm loc}(\mathbb{R}^+)$. Then $\eta = 0$.*

Proof. $\mathscr{A}_t''\Omega$ is dense in $\mathscr{H}_t^{+1}$ and the linear span of $\{W_f(t): f \in L^2_{\rm loc}(\mathbb{R}^+)\}$ is a strongly (in $\mathscr{H}$) dense subalgebra of $\mathscr{A}_t''$. By Kaplansky's density theorem, it follows that the linear span of $\{W_f(t)\,\Omega: f \in L^2_{\rm loc}(\mathbb{R}^+)\}$ is dense in $\mathscr{A}_t''\Omega$ in $\mathscr{H}^{+1}$ and hence is dense in $\mathscr{H}_t^{+1}$. It follows that if $(\eta, W_f(t)\,\Omega)_{\mathscr{H}^{+1}} = 0$ for all $f \in L^2_{\rm loc}(\mathbb{R}^+)$, then $\eta = 0$. Q.E.D.

LEMMA 3.4. *The map $\phi \oplus \bar{\psi} \mapsto \int_0^t \phi(s)\, da_s^* + S\psi(s)\, da_s$ is an isometry from $\mathscr{K}_t \oplus \bar{\mathscr{K}}_t$ into $\mathscr{H}_t^{+1}$, where $\bar{\mathscr{K}}_t$ is the conjugate Hilbert space to $\mathscr{K}_t$.*

Proof. This follows immediately from the isometry property of [1] together with the observation that $\int_0^t \phi(s)\, da_s^*$ and $\int_0^t S\psi(s)\, da_s$ are orthogonal in (both $\mathscr{H}$ and) $\mathscr{H}^{+1}$. This is readily seen if ϕ and ψ are elementary $\mathfrak{P}\Omega$-valued processes; the general case follows by linearity and continuity. Q.E.D.

LEMMA 3.5. *Let $\zeta_t \in \mathscr{H}_t^{+1}$. Then there is ϕ, ψ, in $\mathscr{K}_t$ and $\alpha \in \mathbb{C}$ such that*

$$\zeta_t = \alpha\Omega + \int_0^t \phi(s)\, da_s^* + \int_0^t S\psi(s)\, da_s.$$

Moreover, α, ϕ and ψ are unique.

Proof. By Lemmas 3.2 and 3.3, we deduce that the isometry of Lemma 3.4 is onto the orthogonal complement of Ω in $\mathscr{H}_t^{+1}$. In other words, any element ζ_t of $\mathscr{H}_t^{+1}$ has the required form.

The uniqueness follows from the isometry property [1]. Q.E.D.

Proof of Theorem 3.1. For any $t \geqslant 0$, $\zeta_t \in \mathscr{H}_t^{+1}$ and so, by Lemma 3.5, there exists a unique $\alpha \in \mathbb{C}$, and unique $\phi, \psi \in \mathscr{H}_t$ such that

$$\zeta_t = \alpha\Omega + \int_0^t \phi(s)\, da_s^* + \int_0^t S\psi(s)\, da_s.$$

Let $0 \leqslant r \leqslant t$. Then

$$\zeta_r = P_r \zeta_t = \alpha\Omega + \int_0^r \phi(s)\, da_s^* + \int_0^r S\psi(s)\, da_s.$$

By the uniqueness, it follows that there exists $\alpha \in \mathbb{C}$ and processes ϕ, ψ in $L^2_{\text{loc}}(\mathbb{R}^+, ds; \mathscr{H}^{+1})$ such that

$$\zeta_t = \alpha\Omega + \int_0^t \phi(s)\, da_s^* + \int_0^t S\psi(s)\, da_s$$

for all $t \geqslant 0$, and the theorem is proved. Q.E.D.

References

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